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Integrability in three dimensions: Algebraic Bethe ansatz for anyonic models

Sh. Khachatryan^a, A. Ferraz^b, A. Klümper^c, A. Sedrakyan^{a,b,*}^a *Yerevan Physics Institute, Alikhanian Br. 2, 0036 Yerevan, Armenia*^b *International Institute for Physics, Natal, Brazil*^c *Wuppertal University, Gaußstraße 20, Germany*

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Abstract

We extend basic properties of two dimensional integrable models within the Algebraic Bethe Ansatz approach to $2+1$ dimensions and formulate the sufficient conditions for the commutativity of transfer matrices of different spectral parameters, in analogy with Yang–Baxter or tetrahedron equations. The basic ingredient of our models is the R-matrix, which describes the scattering of a pair of particles over another pair of particles, the quark-anti-quark (meson) scattering on another quark-anti-quark state. We show that the Kitaev model belongs to this class of models and its R-matrix fulfills well-defined equations for integrability. © 2015 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

The importance of 2D integrable models [1–5] in modern physics is hard to overestimate. Being initially an attractive tool in mathematical physics they became an important technique in low dimensional condensed matter physics, capable to reveal non-perturbative aspects in many body systems with great potential of applications. The basic constituent of 2D integrable systems is the commutativity of the evolution operators, the transfer matrices of the models of different spectral parameters. This property is equivalent to the existence of as many integrals of motion as number of degrees of freedom of the model. It appears, that commutativity of transfer matrices

* Corresponding author.

E-mail address: sedrak@nbi.dk (A. Sedrakyan).

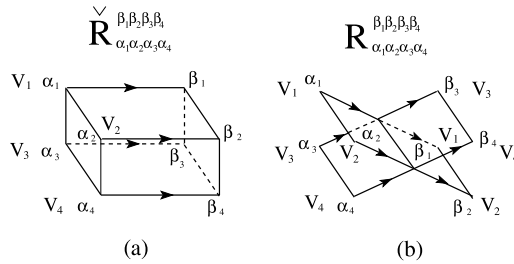


Fig. 1. Four particle R-matrix.

can be ensured by the Yang–Baxter (YB) equations [3–5] for the R-matrix and the integrability of the model is associated with the existence of the solution of YB-equations.

Since the 80s of last century there was a natural desire to extend the idea of integrability to three dimensions [6], which resulted in a formulation of the so-called tetrahedron equation by Zamolodchikov [7]. The tetrahedron equations (ZTE) were studied and several solutions have been found until now [7,8,10,13,14,16–20,22]. However, earlier solutions either contained negative Boltzmann weights or were slight deformations of models describing free particles. Only in a recent work [15] non-negative solutions of ZTE were obtained in a vertex formulation, and these matrices can be served as Boltzmann weights for a 3D solvable model with infinite number of discrete spins attached to the edges of the cubic lattice.

The lack of solutions of tetrahedron equations giving rise to models with finite degrees of freedom at the sites, which one expects in any realistic experimentally relevant situation, rises an immediate question: are there criteria sufficient for integrability of 3D models, that have finite degrees of freedom? This is the precise question we address in this letter.

Although initially the tetrahedron equations were formulated for the scattering matrix S of three infinitely long straight strings in a context of 3D integrability they can also be regarded as weight functions for statistical models. In a Bethe Ansatz formulation of 3D models their 2D transfer matrices of the quantum states on a plane [8,14,17] can be constructed via three particle R-matrix [9,14,21], which, as an operator, acts on a tensorial cube of linear space V , i.e. $R : V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ [11].

Motivated by the desire to extend the integrability conditions in 3D to other formulations we consider a new kind of equations with the R-matrices acting on a quartic tensorial power of linear spaces V

$$\check{R}_{1234} : V_1 \otimes V_2 \otimes V_3 \otimes V_4 \rightarrow V_1 \otimes V_2 \otimes V_3 \otimes V_4, \quad (1)$$

which can be represented graphically as in Fig. 1a.

An important observation in this direction is that Kitaev model [12] can be formulated by the use of this type of R-matrix, which we identify below. Since the model has as much integrals of motion as degrees of freedom, one expects existence of appropriate integrability equations, satisfied by the R-matrix of Kitaev model. Solutions of this integrability equations will lead to the construction of the new type of 3D integrable models, which are essentially different from the Kitaev model.

The main result of this paper is the derivation of a new set of equations – termed as *cubic* equations – that are very different from tetrahedron equations, and define criteria for integrability in 3D. We also show that cubic equations are satisfied by the R-matrix of Kitaev model. We believe that there are many more integrable models in 3D that can be studied within the developed approach.

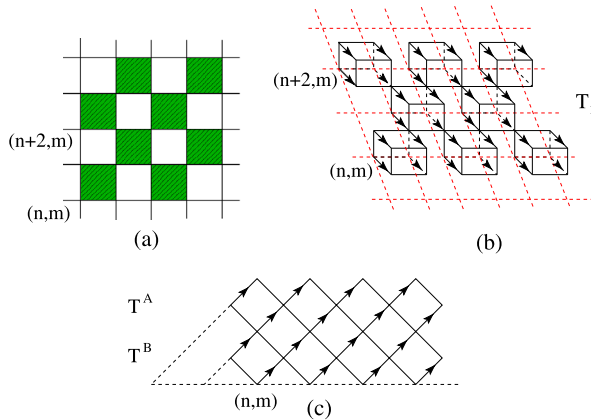


Fig. 2. Transfer matrix T as a product of R -matrices on the plane: a) presents the lattice with chess like structure; b) presents the product of R -matrices arranged in chess like way; c) 2D cut of the product $T_B T_A$.

The R -matrix (1) can be represented also in the form displayed in Fig. 1b, where the final spaces are permuted (V_1 and V_2 with V_3 and V_4 , respectively): $R_{1234} = \check{R}_{1234} P_{13} P_{24}$. Explicitly it can be written as follows

$$R_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^{\beta_1 \beta_2 \beta_3 \beta_4} = \check{R}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^{\beta_3 \beta_4 \beta_1 \beta_2}. \quad (2)$$

Identifying the space $V_1 \otimes V_2$ and $V_3 \otimes V_4$ with the quantum spaces of quark-anti-quark pairs connected with a string one can regard this R -matrix as a transfer matrix for a pair of scattering mesons. Within a terminology used in the algebraic Bethe Ansatz for 1 + 1 integrable models this R -matrix can be viewed also as a matrix, which has two quantum states and two auxiliary states.

The space of quantum states $\Phi_t = \otimes_{(n,m) \in \mathcal{L}} V_{n,m}$ of the system on a plane is defined by a direct product of linear spaces $V_{n,m}$ of quantum states on each site (n, m) of the lattice \mathcal{L} (see Fig. 2a). We fix periodic boundary conditions on both directions: $V_{n,m+L} = V_{n,m}$ and $V_{n+L,m} = V_{n,m}$. The time evolution of this state is determined by the action of the operator/transfer matrix T : $\Phi_{t+1} = \Phi_t T$, which is a product of local evolution operators, R -matrices as follows. First we fix a chess like structure of squares on a lattice \mathcal{L} and associate to each of the black squares a R -matrix $\check{R}_{(n+1,m)(n+1,m+1)(n,m)(n,m+1)}$, which acts on a product of four spaces at the sites. In this way the whole transfer matrix becomes

$$T = \text{Tr} \Pi_{n=1}^{L/2} \left[\Pi_{m=1}^{L/2} \check{R}_{(2n,2m)(2n,2m+1)(2n-1,2m)(2n-1,2m+1)} \cdot \Pi_{m=1}^{L/2} \check{R}_{(2n+1,2m-1)(2n+1,2m)(2n,2m-1)(2n,2m)} \right], \quad (3)$$

where the Trace is taken over states on boundaries. The indices of the R -matrices in the first and second lines of this product just ensure chess like ordering of their action. In Fig. 2b we present this product graphically. First we identify the second pair of states $\langle (2n-1, 2m), (2n-1, 2m+1) \rangle$ (in first row) and $\langle (2n+1, 2m-1), (2n+1, 2m) \rangle$ (in second row) of R -matrices with the corresponding links on the lattice. Then we rotate the box of the R -matrix by $\pi/4$ in order to ensure the correct order for their action in a product. In the same way we define the second list of the transfer matrix, which will act in the order $T_B T_A$. Fig. 2c presents a vertical 2D cut of two lists of the product $T_B T_A$ drawn from the side. The $\pi/4$ rotated lines mark the spaces $V_{n,m}$

attached to sites (n, m) of the lattice. Though transfer matrix (3) is written in \check{R} formalism, it can easily be converted to the product of R -matrices.

The arrangement of R -matrices in the first row (first plane of the transfer matrix T_B) acts on the sites of dark squares of the lattice while R -matrices in the second row (second plane of the transfer matrix T_A) act on the sites of the white squares.

Being an evolution operator the transfer matrix should be linked to time. According to the general prescription [4,5] the transfer matrix $T(u)$ is a function of the so-called spectral parameter u and the linear term H_1 in its expansion $T(u) = \sum_r u^r H_r$ defines the Hamiltonian of the model, while the partition function is $Z = \text{Tr} T^N$. Integrable models should have as many integrals of motion, as degrees of freedom. This property may be reached by considering two planes of transfer matrices with different spectral parameters, $T(u)$ and $T(v)$ and demanding their commutativity $[T(u), T(v)] = 0$, or equivalently demanding the commutativity of the coefficients $[H_r, H_s] = 0$ of the expansion. This means, that all H_r , $r > 1$ are integrals of motion. In 2D integrable models the sufficient conditions for commutativity of transfer matrices are determined by the corresponding YB-equations [3–5].

In order to obtain the analog of the YB equations, which will ensure the commutativity of transfer matrices (3) we use the so-called railway construction. Let us cut horizontally two planes of the R -matrix product of two transfer matrices (on Fig. 2b we present a product of R -matrices for one transfer matrix plane) into two parts and substitute in between the identity

$$\begin{aligned} & \Pi_{m=1}^L id_{(2n+1,m)} id_{(2n,m)} \\ &= \left[\text{Tr} \Pi_{m=1}^L \check{R}_{(2n+1,m)(2n+1,m+1)(2n,m)(2n,m+1)} \right]^{-1} \\ & \cdot \text{Tr} \left[\Pi_{m=1}^L \check{R}_{(2n+1,m)(2n+1,m+1)(2n,m)(2n,m+1)} \right] \end{aligned} \quad (4)$$

which maps two chains of sites, $(2n, m)$, $m = 1 \cdots L$ and $(2n+1, m)$, $m = 1 \cdots L+1$, into itself. The Trace have to be taken by identifying spaces 1 and $L+1$. In this expression we have introduced another set of \check{R} -matrices, called intertwiners, which will be specified below. For further convenience we distinguish $\check{R}_{(2n+1,m)(2n+1,m+1)(2n,m)(2n,m+1)}$ matrices for even and odd values of m marking them as \check{R}_3 and \check{R}_4 respectively. In the left side of Fig. 3 we present one half of the plane of R -matrices together with an inserted chain of $\check{R}_3 \check{R}_4$ as intertwiners. The chain of intertwiners can also be written by R -matrices.

Now let us suggest, that the product of these intertwiners with the first double chain of \check{R} -matrices from the product of two planes of transfer matrices is equal to the product of the same operators written in opposite order. Namely we demand, that

$$\begin{aligned} & \Pi_{m=1}^L \check{R}_{(2n+1,m)(2n+1,m+1)(2n,m)(2n,m+1)} \\ & \cdot \Pi_{m=1}^{L/2} \check{R}_{(2n,2m)(2n,2m+1)(2n-1,2m)(2n-1,2m+1)}(u) \\ & \cdot \Pi_{m=1}^{L/2} \check{R}_{(2n+1,2m)(2n+1,2m+1)(2n,2m)(2n,2m+1)}(v) \\ &= \Pi_{m=1}^{L/2} \check{R}_{(2n,2m)(2n,2m+1)(2n-1,2m)(2n-1,2m+1)}(v) \\ & \cdot \Pi_{m=1}^{L/2} \check{R}_{(2n+1,2m)(2n+1,2m+1)(2n,2m)(2n,2m+1)}(u) \\ & \cdot \Pi_{m=L}^1 \check{R}_{(2n+1,m)(2n+1,m+1)(2n,m)(2n,m+1)}. \end{aligned} \quad (5)$$

Graphically this equation is depicted in Fig. 3. We move the column of intertwiners from the left to the right hand side of the column of two slices of the R -matrix product, simultaneously

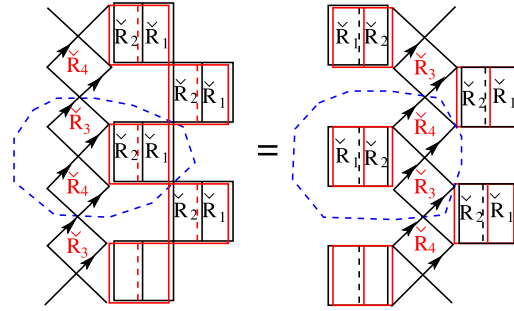


Fig. 3. Reduced set of commutativity conditions for transfer matrices. The dotted subset represents the cubic equations (6).

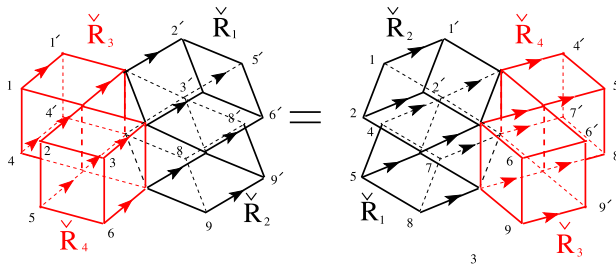


Fig. 4. The set of equations ensuring commutativity of transfer matrices with different spectral parameters. We numerate linear spaces of states V , where R -matrices are acting, by $V_1 \dots V_9$. R -matrices on the left hand side of equation are not acting on space V_7 while R -matrices on the right hand side are not acting on V_3 . We put identity operators in the equation acting on this spaces for the consistency.

changing their order in a column, changing the order of spectral parameters u and v of the slices and demanding their equality. We can use the same type of equality and move the chain of intertwiners further to the right hand side of the next column of the two slices of the \check{R} -matrix product. Then, repeating this operation multiple times, one will approach the chain of inserted \check{R}^{-1} intertwiners inside the Trace from the other side and cancel it. As a result we obtain the product of two transfer matrices in a reversed order of spectral parameters u and v . Hence, the set of equations (5) ensures the commutativity of transfer matrices.

The set of equations (5) can be simplified. Namely, it is easy to see, that the equality can be reduced to the product of only 2 \check{R} -matrices, $\check{R}(u)$ and $\check{R}(v)$ and two intertwiners, R_3 and R_4 . In other words, it is enough to write the equality of the product of \check{R} -matrices from the inside of the dotted line in Fig. 3. Graphically this equation is depicted in Fig. 4.

We see, that in this equation the product of \check{R} -matrices acting on a space $\otimes_{i=1}^9 V_i$ (for simplicity we numerate the spaces from 1 to 9) can be written as

$$\begin{aligned} & \check{R}_{5263}^4(u, v) \check{R}_{4152}^3(u, v) \check{R}_{5689}^2(u) \check{R}_{2356}^1(v) id_7 \\ &= \check{R}_{4578}^1(v) \check{R}_{1245}^2(u) \check{R}_{8596}^3(u, v) \check{R}_{7485}^4(u, v) id_3 \end{aligned} \quad (6)$$

Here we have introduced a short-hand notation for \check{R} -matrices simply by marking the numbers of linear spaces of states, in which they are acting; id_3 and id_7 are identity operators acting on spaces 3 and 7 respectively. Eq. (6) can also easily be written by use of R .

This is the set of equations, sufficient for commutativity of transfer matrices. The same set of equations are sufficient for commuting \check{R} -matrices in the second column in Fig. 1. Equations (6) form an analog of YB equations ensuring the integrability of 3D quantum models. Since they have a form of relations between the cubes of the R-matrix picture (see Fig. 1) we call them cubic equations.

We will show now that the Kitaev model [12] can be described as a model of the prescribed type and its R -matrix fulfills the set of cubic equations (6). The full transfer matrix of Kitaev model is a product $T_A T_B$ of two transfer matrices of type (3) defined by \check{R} -matrices $\check{R}_A = 1 \otimes 1 \otimes 1 \otimes 1 + u \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$ and $\check{R}_B = 1 \otimes 1 \otimes 1 \otimes 1 + u \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z$ respectively. The linear term of the expansion of $T_A T_B$ in the spectral parameter u will produce the Kitaev model Hamiltonian

$$H_{\text{Kitaev}} = \sum_{\text{white plaquettes}} \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x + \sum_{\text{dark plaquettes}} \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z. \quad (7)$$

The integrability of the Kitaev model is trivially clear from the very beginning since all terms in the Hamiltonian defined on white and dark plaquettes commute with each other. The latter indicates, that the number of integrals of motion of the model coincides with its degrees of freedom. However it is important to point out, that the standard Algebraic Bethe Ansatz approach was so far inapplicable to the Kitaev model. The reason is that the corresponding R matrix did not fulfill the tetrahedron equations, which, as in 2D case, would allow to generalize and construct new 3D, similar to Kitaev integrable models. In this paper we have developed the appropriate 3D Algebraic Bethe Ansatz approach, and show, that the Kitaev model belongs to this class of integrability.

Namely, we will show now, that R_A and R_B -matrices of the Kitaev's model fulfill Eq. (6). The explicit form of Eq. (6) by use of indices according to the definition in Fig. 1a reads

$$\begin{aligned} & \check{R}_{\alpha_5 \alpha_2 \alpha_6 \alpha_3}^{\beta_1 \beta_2 \beta_6 \beta_3} \check{R}_{\alpha_4 \alpha_1 \beta_1 \beta_2}^{\gamma_4 \gamma_1 \beta_5 \beta_4} \check{R}_{\beta_5 \beta_6 \alpha_8 \alpha_9}^{\beta_7 \beta_8 \gamma_8 \gamma_9} (v) \check{R}_{\beta_4 \beta_3 \beta_7 \beta_8}^{\gamma_2 \gamma_3 \gamma_5 \gamma_6} (u) \delta_{\alpha_7}^{\gamma_7} \\ &= \check{R}_{\alpha_4 \alpha_5 \alpha_7 \alpha_8}^{\beta_7 \beta_8 \beta_5 \beta_6} (u) \check{R}_{\alpha_1 \alpha_2 \beta_7 \beta_8}^{\gamma_1 \gamma_2 \beta_4 \beta_3} (v) \check{R}_{\beta_6 \beta_3 \alpha_9 \alpha_6}^{\beta_1 \beta_2 \gamma_9 \gamma_6} \check{R}_{\beta_5 \beta_4 \beta_1 \beta_2}^{\gamma_7 \gamma_4 \gamma_8 \gamma_5} \delta_{\alpha_3}^{\gamma_3}, \end{aligned} \quad (8)$$

where $\check{R}^1(u) = \check{R}_A(u)$ and $\check{R}^2(v) = \check{R}_B(v)$. It appears, that the intertwiners

$$\check{R}^4 = \check{R}_A^{-1}(u), \quad \check{R}^3 = \check{R}_B(v) \quad (9)$$

where $\check{R}_A^{-1}(u) = 1 \otimes 1 \otimes 1 \otimes 1 - u \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$ fulfill the cubic equations (8) for any parameters u and v . This can be directly checked both, by a computer algebra program and analytically. The commutativity of transfer matrices $T_A(u)$ with $T_A(v)$ and $T_B(u)$ with $T_B(v)$ is trivial in the Kitaev model.

Summary. We have formulated a class of three dimensional models defined by the R -matrix of the scattering of a two particle state on another two particle state, i.e. a meson–meson type scattering. We derived a set of equations for these R -matrices, which are a sufficient conditions for the commutativity of the transfer matrices with different spectral parameters. These equations differ from the tetrahedron equations, which also ensure the integrability of 3D models, but are based on the R-matrix of 3 particle scatterings. Our set of equations will be reduced to tetrahedron type of equations by considering the two auxiliary spaces in the R-matrix as one (fusion) and

replacing it by one thick line. We showed that the Kitaev model [12] belongs to this class of integrable models. This give rise a hope, that other solutions of integrability equation (6) and (8) with finite degrees of freedom at the sites may be found, which will be non-trivial extensions of Kitaev model.

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